Nonlinear wave equations and dispersive phenomena

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Second order hyperbolic operators in $\mathbb{R} \times \mathbb{R}^n$:

$$\Box_g = g^{ij} \partial_i \partial_j$$

Hyperbolicity: $\{g^{ij}\}_{i,j=\overline{0,n}}$ has signature $(n, 1)$.

Lorentz metric: $(g_{ij}) = (g^{ij})^{-1}$

Surfaces $t = const$ are space-like: $g_{00} < 0$.

Linear initial value problem:

$$\begin{cases} \Box_g u = 0 \qquad in \quad \mathbb{R} \times \mathbb{R}^n \\ u(0, x) = u_0(x) \in H^s(\mathbb{R}^n) \\ \partial_0 u(0, x) = u_1(x) \in H^{s-1}(\mathbb{R}^n) \end{cases}$$

Linear well-posedness: For each initial data $(u_0, u_1) \in H^s \times H^{s-1}$ there exists an unique solution

$$u \in C(\mathbb{R}, H^s(\mathbb{R}^n) \cap C^1(\mathbb{R}; H^{s-1}(\mathbb{R}^n))$$

Fully nonlinear wave equation:

$$F(x, w, \nabla w, \nabla^2 w) = 0$$

$g^{ij} = \dfrac{\partial F(x, w, p, q)}{\partial q_{ij}}$ has signature $(n, 1)$.

Set $v = (w, \nabla w)$ and differentiate to obtain a quasilinear problem:

$$\Box_{g(v,\nabla v)} v = N(v, \nabla v) \qquad (GNLW)$$

Set $u = (v, \nabla v)$ and differentiate:

$$\Box_{g(u)} u = G(u) \nabla u^2 \qquad (NLW)$$

Semilinear wave equation:

$$\Box_g u = G(u) \nabla u^2 \qquad (SLW)$$

The Cauchy problem for (NLW):

$$\begin{cases} \Box_{g(u)} = q^{ij}(u) \partial_i u \, \partial_j u, \\ u(0, x) = u_0(x) \in H^s(\mathbb{R}^n) \\ \partial_0 u(0, x) = u_1(x) \in H^{s-1}(\mathbb{R}^n) \end{cases}$$

Question: For what s is the Cauchy problem locally well-posed in $H^s \times H^{s-1}$?

Definition. The Cauchy problem is locally well-posed in $H^s \times H^{s-1}$ if for each $R > 0$ there exist $T, M > 0$ so that, for each initial data (u_0, u_1) satisfying $\|(u_0, u_1)\|_{H^s \times H^{s-1}} \leq R$,

(i) There is an unique solution

$$u \in C([-T,T]; H^s) \cap C^1([-T,T]; H^{s-1})$$

satisfying

$$\|\nabla u\|_{L^2(L^\infty) \cap L^\infty(H^{s-1})} \leq M$$

(ii) The solution u depends continuously on the initial data (u_0, u_1) in the above topology.

Remarks:

- no Lipschitz dependence on initial data

- uniqueness may require additional regularity properties

Energy estimates

Simplified model:

$$\partial_t^2 u - g^{ij}(u)\partial_i\partial_j u = q^{ij}(u)\partial_i u\,\partial_j u$$

Energy:

$$E_u(u) = \frac{1}{2}\int |\partial_t u|^2 + g^{ij}(u)\partial_i u \partial_j u$$

Energy estimates:

$$\frac{d}{dt}E_u(u) \lesssim \int |\nabla u|^3 \lesssim E(u)\|\nabla u\|_{L^\infty}$$

$$E_u(u)(t) \lesssim E_u(u)(0)e^{\int_0^t \|\nabla u(s)\|_{L^\infty} ds}$$

Level s energy:

$$E_u^s(u) = \sum_{|\alpha|\leq s-1} E_u(\partial^\alpha u)$$

Klainerman:

$$E_u^s(u)(t) \lesssim E_u^s(u)(0)e^{\int_0^t \|\nabla u(s)\|_{L^\infty} ds}$$

No blowup as long as $\nabla u \in L^1 L^\infty$

Regular solutions (large integer s)

a) Uniqueness and weak stability. If u, v solve

$$\partial_t^2 u - g^{ij}(u)\partial_i\partial_j u = 0$$

then $w = u - v$ solves

$$\partial_t^2 w - g^{ij}(u)\partial_i\partial_j w = wA(u,v)\partial^2 v$$

Hence use linear theory to get

$$E_u(u-v)(t) \leq c(u,v)E_u(u-v)(0)$$

with c depending on bounds for $g(u)$ and $\partial^2 v$.

b) Existence. Iterative methods:

$$\partial_t^2 u_{k+1} - g^{ij}(u_k)\partial_i\partial_j u_{k+1} = 0$$

Uniform bounds: If $\nabla u_k \in L^\infty H^{s-1} \cap L^1 L^\infty$ then this is linearly well-posed in $H^s \times H^{s-1}$ so we get $\nabla u_{k+1} \in L^\infty H^{s-1}$. Then $\nabla u_{k+1} \in L^\infty(-T,T;L^\infty)$ from Sobolev.

Weak convergence: from weak stability.

Strong convergence: compactness argument.

Rough solutions (small s)

Energy estimates for noninteger s:

$$E_u^s(v) = E_u(|D|^{s-1}v) \approx \|\nabla u(s)\|_{H^{s-1}}^2$$

Using Kato-Ponce commutator estimates in the equation for $|D|^{s-1}u$ one obtains:

$$E_u^s(u)(t) \lesssim E_u^s(u)(0)e^{\int_0^t \|\nabla u(s)\|_{L^\infty} ds}$$

If the level s energy does not blow up then use weak stability estimates to construct rough solutions as weak limits of smooth solutions.

Classical argument (Hughes-Kato-Marsden):

$$\|\nabla u\|_{L^1 L^\infty} \lesssim T\|\nabla u\|_{L^\infty H^{s-1}} \qquad s > \frac{n}{2} + 1$$

Obstructions to well-posedness:

a) Scaling: $u(t,x) \to u(\lambda t, \lambda x)$, critical index $s_c = \frac{n}{2}$.

b) Blow-up: counterexamples (Lindblad, Alihnac) for $s = \frac{n+5}{4}$.

Dispersive estimates

If v solves $\Box v = 0$ then

$$\|\nabla v\|_{L^4L^\infty} \lesssim \|\nabla v(0)\|_{H^{\frac{n}{2}+\frac{3}{4}}} \qquad n = 2$$

$$\|\nabla v\|_{L^2L^\infty} \lesssim \|\nabla v(0)\|_{H^{\frac{n}{2}+\frac{1}{2}}} \qquad n \geq 3$$

Q: Does this remain true if we replace $\Box$ by $\Box_g$ with $g = g(u)$? If yes, then we have proved the following theorem:

Theorem: (Smith-T. 2001) The Cauchy problem for the nonlinear wave equation is locally well-posed in $H^s \times H^{s-1}$ provided that

$$s > \frac{n}{2} + \frac{3}{4} \qquad n = 2$$

$$s > \frac{n+1}{2} \qquad n = 3, 4, 5$$

This is sharp in dimension n=2, 3.

Difficulty: The metric g is nonsmooth. The best we can hope for is $\nabla g \in L^2L^\infty$.

Strichartz estimates for low regularity metrics

NLW: $s = \frac{n+1}{2} + ?$, $n \geq 3$

			NLW
'90	Kapitanskii Mockeenhaupt- -Seeger-Sogge	g smooth	–
'95	Smith	$g \in C^2,\ n = 2, 3$	–
'95	Smith-Sogge	C^2 is sharp	
'98	Bahouri-Chemin	NLW	$\frac{1}{4}$
'98	Tataru	$\nabla g \in L^2 L^\infty$ ($\frac{1}{4}$ loss)	$\frac{1}{4}$
'99	Tataru	$g \in C^2$	–
'99	Bahouri-Chemin	NLW	$\frac{1}{5+}$
'99	Tataru	$\nabla^2 g \in L^1 L^\infty$ $\nabla g \in L^1 L^\infty$ ($\frac{1}{6}$ loss)	$\frac{1}{6}$
'00	Smith-Tataru	$\frac{1}{6}$ loss is sharp	
'00	Klainerman- –Rodnianski	$NLW,\ \Box_g\ g =$ "nice"	$\frac{1}{7+}$

Weak stability estimates

Proposition: Let u be a solution to NLW which satisfies the conditions in the definition of well-posedness. Let v be another solution to NLW with initial data $(v_0, v_1) \in H^s \times H^{s-1}$ and with regularity $\nabla v \in L^\infty(H^{s-1}) \cap L^2 L^\infty$. Then

$$\|\nabla(u-v)\|_{L^\infty(H^{-\frac{1}{4}})} \leq c_v \|\nabla(u-v)(0)\|_{H^{-\frac{1}{4}}} \quad n = 2$$

$$\|\nabla(u-v)\|_{L^\infty(L^2)} \leq c_v \|\nabla(u-v)(0)\|_{L^2} \quad n \geq 3$$

where c_v depends on $\|\nabla v\|_{L^\infty(H^{s-1}) \cap L^2 L^\infty}$.

$$w = u - v: \ \Box_{g(u)} w = A_1 \nabla w + A_0 w,$$

$$A_1 \approx \nabla(u, v), \ A_0 \approx \nabla^2 v + ...$$

Weak stability $\Longrightarrow$ uniqueness.
Weak stability + compactness $\Longrightarrow$
continuous dependence.

Goal: *Reduce problem to apriori bounds for smooth data.*

Reduction to small, smooth, compactly supported data

Scaling $\Longrightarrow$ reduction to small data.

Finite speed of propagation $\Longrightarrow$ localization $\Longrightarrow$ reduction to compactly supported data.

Proposition: Suppose $\epsilon_3 \ll \epsilon_2 \ll 1$. Assume that the initial data is smooth, supported in $B(0, 10)$ and satisfies

$$\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} \leq \epsilon_3$$

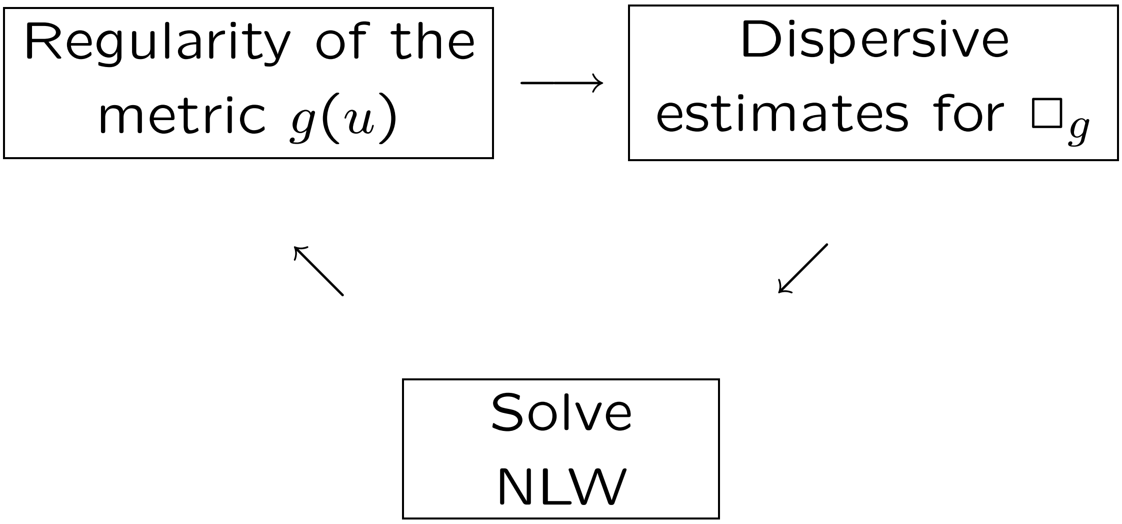
Then NLW has a smooth solution u in $\mathbb{R}^n \times [-1, 1]$ satisfying:

(i) (energy estimate for u) $\|\nabla u\|_{L^\infty(H^{s-1})} \leq \epsilon_2$.

(ii) (dispersive estimate for u) $\|\nabla u\|_{L^2 L^\infty} \leq \epsilon_2$.

(iii) Dispersive estimates hold for the linear operator $\Box_{g(u)}$.

The continuity (bootstrap) argument



Proposition: Let $\epsilon_3 \ll \epsilon_2 \ll \epsilon_1 \ll 1$. Then for solutions u to NLW in $[-1,1]\times\mathbb{R}^n$ the following holds:

$$\left.\begin{aligned} \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} &\leq \epsilon_3 \\ \|\nabla u\|_{L^2L^\infty \cap L^\infty H^{s-1}} &\leq 2\epsilon_2 \\ G(g(u)) &\leq 2\epsilon_1 \end{aligned}\right\} \Longrightarrow$$

$$\left\{\begin{aligned} \|\nabla u\|_{L^2L^\infty \cap L^\infty H^{s-1}} &\leq \epsilon_2 \\ G(g(u)) &\leq \epsilon_1 \end{aligned}\right.$$

The G functional

Work with truncated coefficients on a larger time interval $[-1,1] \to [-2,2]$

$$g(u) \to g(t,x,u) = g_0 + \chi(t)(g(u) - g_0)$$

At $t = -2$ the geometry is flat. For $\theta \in S^{n-1}$ take the plane $\Sigma_{\theta,r}^{-2} = \{x\cdot\theta = -2+r\}$ at time -2 and try to extend it to a smooth characteristic surface $\Sigma_{\theta,r}$ which coincides with $\{x \cdot \theta = t + r\}$ near time -2. If this is not possible then set $G(g) = \infty$. Else write

$$\Sigma_{\theta,r} = \{x_\theta = \phi_{\theta,r}(t, x'_\theta)\} \quad x = (x_\theta, x'_\theta) \quad x_\theta = x\cdot\theta$$

$$G(g) = \sup_{\theta,r} \|\phi_{\theta,r}(t, x'_\theta) - t - r\|_{H^{s+1}}$$

Note that the function $\phi_{\theta,r}$ solves an eikonal equation

$$g^{ij}(t, x'_\theta, \phi_{\theta,r}(t, x'_\theta))\partial_i(x_\theta - \phi_{\theta,r})\partial_j(x_\theta - \phi_{\theta,r}) = 0$$

where coeff. satisfy $g^{ij}(t, x'_\theta, \phi_{\theta,r}) \in H^s$.

The geometry of characteristic “planes”

A characteristic “plane” $\Sigma = \Sigma_{\theta,r}$ is a union of characteristic curves (geodesics). The null vector field tangent to geodesics is denoted by l and is normalized by $l(dt) = 1$. We extend l to a null frame $l, \underline{l}\,, e_a$ so that l, e_a span $T\Sigma$, $e_a(dt) = 0$ and

$$\langle l, l \rangle = \langle \underline{l}\,, \underline{l}\, \rangle = \langle l, e_a \rangle = \langle \underline{l}\,, e_a \rangle = 0,$$

$$\langle l, \underline{l}\, \rangle = 2, \quad \langle e_a, e_b \rangle = \delta_{ab}$$

The interesting part of the second fundamental form of Σ is the deformation tensor $\chi_{ab} = \langle \nabla_{e_a} l, e_b \rangle$. Essentially we have

$$\Sigma \text{ is } H^{s+1} \quad \Leftrightarrow \quad \chi_{ab} \in H^{s-1}(\Sigma)$$

The transport equation for χ_{ab} is the Raychadhuri equation,

$$l\chi_{ab} = R_{albl} + \chi_{ab}\chi_{bc} + \text{“good terms”}$$

Lemma: We may write $R_{albl} = l(f_2) + f_1$ with

$$\|f_2\|_{L^2_t H^{s-1}_{x'}(\Sigma)} + \|f_1\|_{L^1_t H^{s-1}_{x'}(\Sigma)} \lesssim \epsilon_2\,,$$

Proof: We express the curvature tensor as

$R_{albl} = e_a e_b g_{ll} + l l g_{ab} - l e_a g_{lb} - l e_b g_{la} +$ "good terms"

Furthermore, $\Box_g = e_a e_a - l\,\underline{l} +$ "good". Then we use the equation for g, $\Box_g g =$ "good" to write

$$e_a e_b g_{ll} = l(e_a e_b (e_a e_a)^{-1})\,\underline{l}\,g_{ll} + \text{"good"}$$

Here all the "good" terms contribute only to f_2.

Important observation: This computation can be done only for some components of the curvature tensor. Consequently, we have a good control of the geometry of individual characteristic "planes", but not of the relative geometry of neighboring characteristic "planes". Badly behaved quantities are $\nabla_{\underline{l}}\, l$ (characteristic "planes" can slide one on top of another) and σ, defined by $l(\ln \sigma) = \langle \nabla_l l, \underline{l}\,\rangle$, which measures the distance between parallel characteristic "planes".

The geometry of cones

We show that characteristic "cones" starting at arbitrary points in the space-time are well behaved. We study the same Raychadhuri equation,

$$l\chi_{ab} = R_{albl} + \chi_{ab}\chi_{bc} + \text{“good terms”}$$

but with a weaker Lemma which only uses pointwise bounds,

Lemma: We may write $R_{albl} = l(f_2) + f_1$ with

$$\|f_2\|_{L^2 L^\infty} + \|f_1\|_{L^1 L^\infty} \lesssim \epsilon_2\,,$$

and with a different boundary condition at the tip,

$$\lim_{t \to t_0} (t - t_0)\chi = I_{n-1}$$

Consequence: We get a good control of the angle between intersecting characteristic planes,

$$l_\theta - l_\omega = \theta - \omega + o(|\theta - \omega|)$$

The paradifferential decomposition

To solve the linear equation $\Box_g v = 0$ we localize in frequency. If v is at frequency λ then we are allowed to truncate g at frequency $\ll \lambda$. The dispersive estimates for the linear equation reduce to

Proposition: For each initial data (v_0, v_1) localized at frequency λ there is a function v_λ localized at frequency λ so that

(i) (matching the initial data)

$$v_\lambda(0) = v_0, \qquad v_{\lambda t}(0) = v_1$$

(ii) (approximate solution)

$$\|\Box_{g_{<\lambda}} v_\lambda\|_{L^1 L^2} \ll \|v_0\|_{H^1} + \|v_1\|_{L^2}$$

(iii) (dispersive estimates)

$$\|\nabla v_\lambda\|_{L^2 L^\infty} \lesssim \lambda^{\frac{n-1}{2}+\delta} (\|v_0\|_{H^1} + \|v_1\|_{L^2})$$

Wave packets

Wave packets are highly localized approximate solutions to the wave equation. Given a null bicharacteristic (geodesic) γ on a characteristic "plane" $\Sigma = \{x_1 = \phi(t, x')\}$, we can construct such a wave packet for $\Box_{g_{<\lambda}}$ which is (a bump) supported within λ^{-1} of Σ and $\lambda^{-\frac{1}{2}}$ of γ. Its support is contained in a thin curved parallelepiped (slab) of size $1 \times \lambda^{-1} \times (\lambda^{-\frac{1}{2}})^{n-1}$. One possible way of obtaining it is

$$u = \lambda^{\frac{n-7}{4}} T_\lambda \left(\chi(\lambda^{\frac{1}{2}}(x - \gamma(t)) \delta_{x_1 = \phi(t,x')} \right)$$

where T_λ is a mollifier with compactly supported kernel acting on the λ^{-1} scale. Then

$$\|\nabla u(t)\|_{L^2_x} \approx 1, \qquad \|\Box_{g_{<\lambda}} u\|_{L^2_{x,t}} \ll 1$$

The parametrix

To construct the parametrix we work with wave packets corresponding to a discrete set of directions θ which are $\lambda^{-\frac{1}{2}}$ separated. For each θ we consider a locally finite partition of the space into θ-slabs.

Proposition: For initial data $(S_\lambda v_0, S_\lambda v_1)$ at time -2 there is an approximate solution v_λ which is a superposition of normalized wave packets

$$v_\lambda = S_\lambda \sum_T a_T u_T \qquad \sum_T a_T^2 \approx \|S_\lambda v_0\|_{H^1} + \|S_\lambda v_1\|_{L^2}$$

The orthogonality estimates for wave packets:

$$\|\nabla v_\lambda(t)\|_{L^2} \lesssim \left(\sum a_T^2\right)^{\frac{1}{2}} \qquad \|\Box_{g_{<\lambda}} v_\lambda\|_{L^1 L^2} \ll \left(\sum a_T^2\right)^{\frac{1}{2}}$$

use the fact that we control the angle at which slabs intersect.

The dispersive estimates

We need L^2L^∞ estimates for superpositions of wave packets. This uses only the size of wave packets, and not the oscillation. Our estimates have a log loss:

Proposition: We have

$$\|\sum_T a_T \chi_T\|_{L^4(L^\infty)} \lesssim (\ln\lambda)^2 \left(\sum a_T^2\right)^{\frac{1}{2}} \qquad n = 2$$

$$\|\sum_T a_T \chi_T\|_{L^2(L^\infty)} \lesssim \lambda^{\frac{n-3}{4}} (\ln\lambda)^3 \left(\sum a_T^2\right)^{\frac{1}{2}} \qquad n \geq 3$$

The proof is quite elementary. It uses a counting argument, which requires very little geometric information about the wave packets.

The overlapping lemma

This is the main ingredient in the proof of the dispersive estimates:

Proposition: For all points $P_1 = (t_1, x_1)$ and $P_2 = (t_2, x_2)$ in space-time the number of slabs of scale λ that contain both P_1 and P_2 is bounded by

$$N_\lambda(P_1, P_2) \lesssim \lambda^{\frac{n-3}{2}} |t_1 - t_2|^{-1} \qquad n \geq 3$$

$$N_\lambda(P_1, P_2) \lesssim |t_1 - t_2|^{-\frac{1}{2}} \qquad n = 2$$

This result is the same as in the constant coefficient case. However, its proof is considerably more delicate since we do not control all aspects of the geometry.

Further questions:

- Extend this result to dimension 6 and higher.

- Remove the gap between the positive result and the counterexamples in dimension 4 and higher.

- Improve the result for equations with special structure (null -condition)

Preprint:

`http://www.math.berkeley.edu/~tataru`